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LARGE ELASTIC DEFORMATIONS OF ISOTROPIC MATERIALS
X. REINFORCEMENT BY INEXTENSIBLE CORDS

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CONTENTS

The theory of large elastic deformations of incompressible, isotropic materials developed in previous papers of this series is employed to examine some simple deformations of elastic bodies reinforced with cords. The cords are assumed to be thin, flexible and inextensible, and to lie parallel and close together in smooth surfaces in the undeformed body, which is thus divided into sections by boundary surfaces which are inextensible in certain directions. In the simple problems considered, the cords impose relationships upon the parameters which specify the deformation.

The following examples are examined from this point of view:

(i) the pure homogeneous strain of a thin uniform sheet containing a double layer of cords lying in a plane midway between its major surfaces;

(ii) the combined pure homogeneous strain and flexure of a cuboid containing a double layer of cords lying in a plane parallel to a pair of opposite faces, the two sets of cords being unsymmetrically disposed in this plane with respect to the remaining faces of the cuboid, and the symmetrical case being obtained from this by a suitable choice of constants;

(iii) the combined extension and flexure of a thin rectangular sheet with two sets of cords placed symmetrically in a plane parallel to its major surfaces, the problem being considered as a limiting case of (ii);

(iv) the simultaneous extension, inflation and torsion of a cylindrical tube containing one or two sets of cords lying in helical paths concentric with the axis of the cylinder.

In all cases, relations are obtained for the determination of the tensions in the cords in terms of the applied forces and the parameters which define the deformation.

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1. Introduction

In many applications of rubber-like materials, reinforcement is provided by the introduction of layers of cords which have a much higher modulus of elasticity than that of the surrounding material. Such reinforcement forms a feature, for example, of the construction of such common articles as pneumatic tyres and fire hose, in which it is necessary to restrict the magnitude of deformation in certain directions and to give added strength to the composite body.

In previous papers of this series, a theory of large elastic deformation has been developed by Rivlin (1948 *a*, *b*, 1949 *a*, *b*) which is applicable to unreinforced ideal rubber-like materials, for which it is possible to introduce the assumptions of isotropy and incompressibility. The elastic properties of such materials may be expressed in terms of a strain-energy function W which is a function of two invariants of strain I_1 and I_2 defined by

$$I_1 = \sum_{i=1}^{3} \lambda_i^2, \quad I_2 = \sum_{i=1}^{3} \lambda_i^{-2},$$

λ_i being the principal extension ratios at any point of the deformed body. Employing the results of this theory, the form of W for a natural rubber vulcanizate has been investigated experimentally by Rivlin & Saunders (1951).

The assumption of incompressibility imposes a constraint upon the deformation which makes possible the solution of a number of problems without any restriction either upon the form of strain-energy function or upon the magnitude of the deformation. Owing to the non-linearity of the equations which must be employed, however, the successful treatment of such problems has depended upon the imposition of a general restriction upon the form of the deformation, so that, in the subsequent analysis, partial differential equations can be avoided. By employing the linear form of strain-energy function postulated by Mooney (1940) solutions of somewhat more general problems have been obtained (Rivlin 1949 *b*; Adkins 1954, 1955), but even with this limitation on the form of W the difficulties inherent in the treatment of non-linear partial differential equations, in general, remain.

When the elastic material is reinforced with sets of cords, further assumptions must be made in order to examine the deformation of the composite body. In the present paper we assume the cords to be thin, flexible and inextensible, and to lie parallel and close together in smooth surfaces in the undeformed body. The cords thus introduce into the elastic body thin, flexible surfaces which are inextensible in certain directions. Each layer may then be regarded as a boundary surface for the elastic material on either side, and the elastic problem for each section of the material into which the body is divided by the cords considered separately. For each such region, the difficulties inherent in the consideration of large deformations of unreinforced materials again arise, and attention is therefore confined to problems of the type already solved for such materials. The equilibrium of each layer of cords under the action of the forces due to the adjoining elastic material may be considered by the usual methods employed for thin shells. This approach has already been employed by Adkins (1951) in considering reinforced cylindrical tubes and thin sheets, and the present paper includes a more general treatment of these problems.

The simplest problem, considered in §§2 and 3, is that of the homogeneous deformation of a uniform thin plane sheet of elastic material, which is reinforced by means of a single

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layer containing two sets of straight cords, the cords of each set being parallel to each other and intersecting those of the other set at a constant angle. It follows from geometrical considerations that the principal directions of strain bisect the angles between the cords, the presence of which implies a relationship between the principal extension ratios additional to that already resulting from the incompressibility condition. If the deformation is produced by means of simple tensile forces uniformly distributed along a pair of opposite edges in a direction bisecting the angle between the cords, it is possible for the tensions in the individual cords to become negative at a critical value of the extension ratio which depends upon the angle which the cords make with the direction of the force. This would suggest that the equilibrium then becomes unstable, but this aspect is not considered in the present paper.

The flexure of a cuboid containing a similar double layer of cords placed in a plane of flexure is examined in §§4 to 8. It is assumed initially that the cords are placed unsymmetrically with respect to the axis of flexure, and that a pure homogeneous deformation with two of its principal directions bisecting the angles between the cords is applied prior to flexure, the results obtained by Rivlin (1949*a*, *b*) for an unreinforced cuboid being generalized to include this unsymmetrical case. The symmetrical case is considered in §8, and the results thus obtained employed to examine the flexure of a uniform thin plane sheet, considered as the limiting case of a cuboid, in §9. By this means it is possible to infer that when the plane of the cords does not lie exactly midway between the major surfaces of the undeformed sheet, a simple tensile force is sufficient to produce flexure, in addition to extending the sheet in the direction of the force. The axis of the flexure is parallel to the direction of the applied force, but the sense in which the bending occurs depends upon the angle between the cords and the magnitude of the extension. The relationship which must hold between these quantities for zero flexure is identical with that obtained in §3 for the cords to be unstressed in the case of simple extension.

The simultaneous extension, inflation and torsion of a reinforced cylindrical tube of material is examined in the final sections of the paper. The cords are assumed to take the form of helices lying in cylindrical surfaces co-axial with the boundaries of the elastic cylinder, and the geometrical constraints which are thus imposed ensure that a continuous deformation of this type is only possible if there are not more than two such independent sets of cords. When the two sets of cords are symmetrically disposed in the same layer and the tube extended by means of a simple longitudinal force alone, it is possible for the tensions in the cords to become negative, the relationship between the critical extension ratio at which this occurs and the angle between the cords again being identical with that obtained in §3 for the case of simple extension.

PURE HOMOGENEOUS DEFORMATION OF A THIN SHEET

2. The general case

We shall suppose the undeformed body to be a thin plane sheet of highly elastic, isotropic, incompressible material, bounded in the rectangular Cartesian co-ordinate system (x, y, z) by the planes $z = \pm\frac{1}{2}h$. This sheet is reinforced by means of two sets of thin, straight inextensible cords lying in the plane $z = 0$, the cords of each set being parallel to each other and intersecting those of the other set at a constant angle.

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The sheet is subjected to a pure homogeneous strain with principal extension ratios $\lambda_1, \lambda_2, \lambda_3$ in the directions x, y, z respectively. From geometrical considerations it is readily seen that this deformation is possible only if the cords are placed symmetrically with respect to the axes, so that the principal directions of strain bisect the angles between them. Thus, if an element of length ds in the undeformed sheet with direction cosines (l, m, n) attains a length ds' after deformation we have

$$\left(\frac{ds'}{ds}\right)^2 = \lambda_1^2 l^2 + \lambda_2^2 m^2 + \lambda_3^2 n^2. \tag{2·1}$$

If the cords lie in the directions $(l_1, m_1, 0)$ and $(l_2, m_2, 0)$ so that $ds'/ds = 1$ in these directions, we obtain from (2·1)

$$\lambda_1^2 l_1^2 + \lambda_2^2 m_1^2 = 1, \quad \lambda_1^2 l_2^2 + \lambda_2^2 m_2^2 = 1,$$

and since

$$l_1^2 + m_1^2 = l_2^2 + m_2^2 = 1,$$

these relations yield

$$l_1^2 = l_2^2 = (1 - \lambda_2^2)/(\lambda_1^2 - \lambda_2^2),$$

or

$$l_1 = \pm l_2, \quad m_1 = \pm m_2,$$

which is the required condition of symmetry.

If the cords of each set are inclined at angles $\pm\alpha$, $\pm\beta$ to the x-axis before and after deformation respectively, we have

$$\cos\beta = \lambda_1 \cos\alpha, \quad \sin\beta = \lambda_2 \sin\alpha, \tag{2·2}$$

and

$$\lambda_1^2 \cos^2\alpha + \lambda_2^2 \sin^2\alpha = 1. \tag{2·3}$$

Since, for an incompressible material,

$$\lambda_1 \lambda_2 \lambda_3 = 1, \tag{2·4}$$

it is evident that any one of the quantities $\lambda_1, \lambda_2, \lambda_3$ or β is sufficient to specify the deformation completely. The strain invariants I_1, I_2 defined in § 1 are not therefore independent, and the strain-energy function W could be considered as a function of a single invariant. A similar result has been obtained in other instances where the deformation is subject to a general constraint by Adkins, Green & Shield (1953) and by Adkins (1954), but no great simplification results from this consideration in the present instance.

We shall now suppose the deformation to be produced by a system of forces acting in the plane of the sheet and applied uniformly around its edges so that the stress resultants acting at any point of the sheet are

(i) T_1, S_1, in the positive directions of x, y respectively across a line parallel to the y-axis, and
(ii) T_2, S_2 in the positive directions of y, x respectively across a line parallel to the x-axis,

as shown in figure 1, these forces being measured per unit length of edge of the deformed sheet.

We may consider the elastic material and the system of cords separately and write

$$T_1 = T_1' + T_1'', \quad T_2 = T_2' + T_2'', \tag{2·5}$$

where (T_1', T_2') are the forces required to deform the elastic material alone, and (T_1'', T_2'') are due to tensions in the cords. Since the force system is specified with respect to the principal directions of strain, S_1 and S_2 arise entirely from the tensions in the cords. T_1', T_2'

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are given by the expressions found by Adkins & Rivlin (1952) in considering large deformations of thin shells. Thus

$$\left.\begin{aligned} T_1' &= 2\lambda_3 h(\lambda_1^2-\lambda_3^2)\left(\frac{\partial W}{\partial I_1}+\lambda_2^2\frac{\partial W}{\partial I_2}\right),\\ T_2' &= 2\lambda_3 h(\lambda_2^2-\lambda_3^2)\left(\frac{\partial W}{\partial I_1}+\lambda_1^2\frac{\partial W}{\partial I_2}\right).\end{aligned}\right\} \tag{2·6}$$

Let us assume that the cords making an initial angle $+\alpha$ with the x-axis are spaced a distance d_1 apart before deformation and that each cord carries a tension τ_1 when the sheet is deformed, the corresponding quantities for the other set of cords being denoted by d_2, τ_2 respectively. Lines in the x and y directions are intersected initially by $\sin\alpha/d_1$, $\cos\alpha/d_1$ cords of the first set per unit length respectively, these quantities being changed to

$$\sin\alpha/(\lambda_1 d_1),\quad \cos\alpha/(\lambda_2 d_1)$$

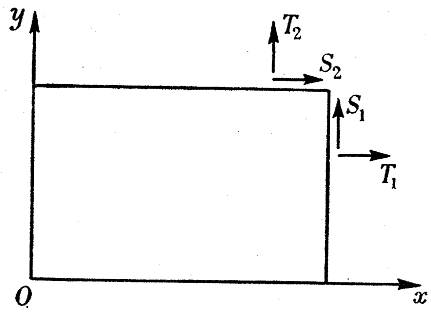

FIGURE 1

in the final configuration. Applying similar considerations to the second set of cords, and remembering the relations (2·2) for the angle β at which cords are inclined to the x-axis after deformation, we have by a simple resolution of forces

$$\left.\begin{aligned} T_1'' &= (\lambda_1/\lambda_2)(\sigma_1+\sigma_2)\cos^2\alpha,\\ T_2'' &= (\lambda_2/\lambda_1)(\sigma_1+\sigma_2)\sin^2\alpha,\\ S_1 &= S_2 = \tfrac{1}{2}(\sigma_1-\sigma_2)\sin 2\alpha = S,\end{aligned}\right\} \tag{2·7}$$

where $$(\sigma_1,\sigma_2) = (\tau_1/d_1, \tau_2/d_2).$$

Combining (2·5), (2·6) and (2·7) we have, finally,

$$\left.\begin{aligned} T_1 &= 2\lambda_3 h(\lambda_1^2-\lambda_3^2)\left(\frac{\partial W}{\partial I_1}+\lambda_2^2\frac{\partial W}{\partial I_2}\right)+\frac{\lambda_1}{\lambda_2}(\sigma_1+\sigma_2)\cos^2\alpha,\\ T_2 &= 2\lambda_3 h(\lambda_2^2-\lambda_3^2)\left(\frac{\partial W}{\partial I_1}+\lambda_1^2\frac{\partial W}{\partial I_2}\right)+\frac{\lambda_2}{\lambda_1}(\sigma_1+\sigma_2)\sin^2\alpha.\end{aligned}\right\} \tag{2·8}$$

3. The case of simple extension

When the deformation is produced by means of a tensile force T_1 applied in the direction of the x-axis, $T_2 = S = 0$ and equations (2·7) and (2·8) then yield

$$\sigma_1 = \sigma_2 = -\frac{h}{\lambda_2^2}(\lambda_2^2-\lambda_3^2)\left(\frac{\partial W}{\partial I_1}+\lambda_1^2\frac{\partial W}{\partial I_2}\right)\operatorname{cosec}^2\alpha = \sigma \quad \text{(say)}, \tag{3·1}$$

and

$$T_1 = 2h\frac{\lambda_1}{\lambda_2}\left\{[(1-\cot^2\alpha)+\lambda_3^4(\lambda_1^2\cot^2\alpha-\lambda_2^2)]\frac{\partial W}{\partial I_1}+[(\lambda_2^2-\lambda_1^2\cot^2\alpha)+\lambda_3^4(\lambda_1^4\cot^2\alpha-\lambda_2^4)]\frac{\partial W}{\partial I_2}\right\}. \tag{3·2}$$

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From (3·1) we see that, since $\partial W/\partial I_1 + \lambda_1^2 \partial W/\partial I_2$ must be positive from physical considerations, σ is positive if

$$\lambda_2^2 - \lambda_3^2 < 0. \tag{3·3}$$

Also, from (2·3) and (2·4) we may obtain

$$\lambda_2^2 - \lambda_3^2 = \frac{(\lambda_1^6 - 1)\{\cos^2\alpha - 1/(\lambda_1^2 - \lambda_1 + 1)\}\{\cos^2\alpha - 1/(\lambda_1^2 + \lambda_1 + 1)\}}{\lambda_1^2 \sin^2\alpha (1 - \lambda_1^2 \cos^2\alpha)}. \tag{3·4}$$

Now, since we have assumed a state of simple extension $\lambda_1 > 1$. Also, from (2·3), since $\lambda_2 \sin\alpha$ is real, $1 - \lambda_1^2 \cos^2\alpha$ must be positive and hence, since $\lambda_1 > 1$, $\cos^2\alpha - 1/(\lambda_1^2 - \lambda_1 + 1)$ must be negative. The sign of σ will therefore be that of $\cos^2\alpha - 1/(\lambda_1^2 + \lambda_1 + 1)$. Hence, remembering (2·3), it follows that provided α is taken as lying between 0 and $\frac{1}{2}\pi$, σ is positive if $\cos^{-1}(1/\lambda_1) < \alpha < \cos^{-1}(\lambda_1^2 + \lambda_1 + 1)^{-\frac{1}{2}}$ and negative if $\cos^{-1}(\lambda_1^2 + \lambda_1 + 1)^{-\frac{1}{2}} < \alpha < \frac{1}{2}\pi$. Thus, for any given value of α, there is a critical value of the extension ratio λ_1, above which the tension in the cords becomes positive, and below which it is negative. This critical value of λ_1 increases as α is increased.

THE FLEXURE OF A CUBOID CONTAINING INEXTENSIBLE CORDS

4. Geometrical considerations

The simple flexure of a cuboid of isotropic incompressible material has been considered by Rivlin (1949 *a*, *b*). In this part of the paper we shall extend the theory to include the case where the cuboid contains a layer of thin inextensible cords and a homogeneous deformation is superposed on the simple flexure, the principal axes of this deformation being unsymmetrically placed with respect to the axis of flexure.

We assume the undeformed elastic body to be a cuboid bounded in the rectangular Cartesian co-ordinate system (x, y, z) by the planes $x = a_1$, $x = a_2$, $y = \pm b$, $z = \pm c$, and reinforced by means of a layer of thin inextensible cords lying in the plane $x = a_0$. This layer consists of two sets of parallel straight cords which make angles $\pm\alpha$ with the z'-axis of a second rectangular Cartesian co-ordinate system (x', y', z') related to the system (x, y, z) by the scheme

$$\begin{array}{cccc} & x & y & z \\ x' & 1 & 0 & 0 \\ y' & 0 & l & m \\ z' & 0 & -m & l \end{array} \tag{4·1}$$

The origin and x'-axis of the system (x', y', z') thus coincide with the origin and x-axis of the system (x, y, z) and the y'-axis of the former system makes an angle ϕ with the y-axis of the latter, measured anticlockwise, where $(l, m) = (\cos\phi, \sin\phi)$. The cords making an angle $+\alpha$ with the z'-axis are spaced a distance d_1 apart initially, and each carries a tension τ_1 when the body is deformed. Quantities d_2, τ_2 are similarly defined for the other set of cords, and we write $(\tau_1/d_1, \tau_2/d_2) = (\sigma_1, \sigma_2)$ to conform with the notation of § 2.

We may regard the resultant deformation to which the cuboid is subjected to have been produced in two stages as follows:

(i) a pure homogeneous strain with principal extension ratios $(\lambda_1, \lambda_2, \lambda_3)$ in the directions (x', y', z');

(ii) a simple flexure, symmetrical with respect to the x-axis, in which each plane normal to the x-axis becomes in the deformed state, part of a curved surface of a cylinder having the

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z-axis as axis, and in which planes which were normal to the y-axis after the pure homogeneous deformation (i) become planes containing the z-axis. During this flexure, elements of length lying in the layer of cords remain unchanged.

In the deformation (i), a point initially at (x', y', z') in the (x', y', z') system moves to (ξ', η', ζ') in the same co-ordinate system where

$$(\xi', \eta', \zeta') = (\lambda_1 x', \lambda_2 y', \lambda_3 z'). \tag{4·2}$$

If the initial and final co-ordinates of this point in the (x, y, z) co-ordinate system are (x, y, z) and (ξ'', η'', ζ'') respectively, we may transform (4·2) by means of (4·1) to obtain

$$(\xi'', \eta'', \zeta'') = (\lambda_1 x, K_1 y + Lz, Ly + K_2 z), \tag{4·3}$$

where
$$K_1 = l^2\lambda_2 + m^2\lambda_3, \quad K_2 = m^2\lambda_2 + l^2\lambda_3, \quad L = lm(\lambda_2 - \lambda_3). \tag{4·4}$$

From (4·4) and (2·4) we have the relations

$$\left.\begin{aligned} K_1 + K_2 = \lambda_2 + \lambda_3, \quad K_1K_2 - L^2 = 1/\lambda_1, \\ K_1^2 + L^2 = l^2\lambda_2^2 + m^2\lambda_3^2, \quad K_2^2 + L^2 = m^2\lambda_2^2 + l^2\lambda_3^2, \end{aligned}\right\} \tag{4·5}$$

which will be used subsequently. Since the cords are inextensible, from (2·3) we have

$$\lambda_2^2 \sin^2\alpha + \lambda_3^2 \cos^2\alpha = 1, \tag{4·6}$$

so that $K_1^2 + L^2 = 1$ if $\phi = \pm(\frac{1}{2}\pi - \alpha)$ and $K_2^2 + L^2 = 1$ if $\phi = \pm\alpha$.

For the simple flexure defined by (ii) we may employ the results obtained by Rivlin (1949 *a*, § 2) for incompressible materials and write

$$(\xi, \eta, \zeta) = \{(2A\xi'' + B)^{\frac{1}{2}} \cos(\eta''/A), (2A\xi'' + B)^{\frac{1}{2}} \sin(\eta''/A), \zeta''\}, \tag{4·7}$$

where (ξ, η, ζ) are the final co-ordinates of the point (x, y, z) in the co-ordinate system (x, y, z), and A and B are constants. If, therefore, we choose a cylindrical polar co-ordinate system (r, θ, z) so that

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{4·8}$$

and consider that as a result of the combined deformations (i) and (ii) the point (r, θ, z) is displaced to (ρ, ϑ, ζ), where

$$\xi = \rho\cos\vartheta, \quad \eta = \rho\sin\vartheta,$$

we have, from (4·3) and (4·7),

$$\rho = (2A\lambda_1 x + B)^{\frac{1}{2}}, \quad \vartheta = (K_1 y + Lz)/A, \quad \zeta = Ly + K_2 z. \tag{4·9}$$

Let the planes $x = a_0$, $x = a_1$, $x = a_2$ in the undeformed body become parts of cylinders of radii r_0, r_1, r_2 in the final configuration. Without loss of generality we may assume $a_1 \geq a_0 \geq a_2 > 0$ and $r_1 \geq r_0 \geq r_2 > 0$. Since elements of length in the layer of cords are unchanged in length by the simple flexure, we have $d\eta'' = r_0 d\vartheta$ and then from (4·3) and (4·9) $A = r_0$. The relations (4·9) then yield

$$\left.\begin{aligned} B = r_0(r_0 - 2\lambda_1 a_0), \quad \rho = \{r_0[r_0 + 2\lambda_1(x - a_0)]\}^{\frac{1}{2}}, \\ r_i^2 = r_0\{r_0 + 2\lambda_1(a_i - a_0)\} \quad (i = 1, 2). \end{aligned}\right\} \tag{4·10}$$

Remembering (2·4), (4·4), (4·6) and (4·8), it follows from (4·9) that the deformation may be specified completely by the radius of flexure r_0 of the layer of cords and any one of the principal extension ratios $\lambda_1, \lambda_2, \lambda_3$ in this layer.

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5. The stress-strain relations and equations of equilibrium

Formulae in cylindrical polar co-ordinates for large elastic deformations of isotropic incompressible materials have been given by Rivlin (1949 *b*, §2). Thus if (r,θ,z), (ρ,ϑ,ζ) are the initial and final co-ordinates respectively of any point of the elastic body in the cylindrical polar co-ordinate system (r,θ,z), and quantities $a_{ij} (i,j=1,2,3)$ are defined by

$$a_{ij}=\begin{pmatrix}\rho_r, & \rho_\theta/r, & \rho_z\\ \rho\vartheta_r, & \rho\vartheta_\theta/r, & \rho\vartheta_z\\ \zeta_r, & \zeta_\theta/r, & \zeta_z\end{pmatrix}, \tag{5·1}$$

the incompressibility condition takes the form

$$\det a_{ij}=1, \tag{5·2}$$

and the strain invariants I_1 and I_2 are given by

$$I_1=a_{ij}a_{ij}, \quad I_2=A_{ij}A_{ij}, \tag{5·3}$$

where we have used the summation convention for repeated suffixes, and A_{ij} denotes the minor of a_{ij} in $\det a_{ij}$. The stress components can now be written

$$t_{ij}=2\left\{a_{ik}a_{jk}\frac{\partial W}{\partial I_1}-A_{ik}A_{jk}\frac{\partial W}{\partial I_2}\right\}+p\delta_{ij}, \tag{5·4}$$

where p is an arbitrary hydrostatic pressure, δ_{ij} is the Kronecker delta and the stress components $(t_{\rho\rho}, \ldots, t_{\rho\vartheta})$ are denoted by $(t_{11}, \ldots, t_{12})$ respectively.

Introducing (4·9), with $A=r_0$, into these relations and making use of (4·8) and (4·5), we obtain

$$\left.\begin{aligned}a_{ij}&=\begin{pmatrix}(r_0\lambda_1/\rho)\cos\theta, & -(r_0\lambda_1/\rho)\sin\theta, & 0\\ (\rho K_1/r_0)\sin\theta, & (\rho K_1/r_0)\cos\theta, & \rho L/r_0\\ L\sin\theta, & L\cos\theta\,, & K_2\end{pmatrix},\\ A_{ij}&=\begin{pmatrix}\rho\cos\theta/(r_0\lambda_1), & -\rho\sin\theta/(r_0\lambda_1), & 0\\ (r_0\lambda_1K_2/\rho)\sin\theta, & (r_0\lambda_1K_2/\rho)\cos\theta, & -r_0\lambda_1L/\rho\\ -\lambda_1L\sin\theta, & -\lambda_1L\cos\theta, & K_1\lambda_1\end{pmatrix},\end{aligned}\right\} \tag{5·5}$$

$$\left.\begin{aligned}I_1&=\frac{r_0^2\lambda_1^2}{\rho^2}+\frac{\rho^2}{r_0^2}(l^2\lambda_2^2+m^2\lambda_3^2)+m^2\lambda_2^2+l^2\lambda_3^2,\\ I_2&=\frac{\rho^2}{r_0^2\lambda_1^2}+\frac{r_0^2}{\rho^2}\left(\frac{l^2}{\lambda_2^2}+\frac{m^2}{\lambda_3^2}\right)+\frac{m^2}{\lambda_2^2}+\frac{l^2}{\lambda_3^2},\end{aligned}\right\} \tag{5·6}$$

and

$$\left.\begin{aligned}t_{\rho\rho}&=2\left\{\frac{r_0^2\lambda_1^2}{\rho^2}\frac{\partial W}{\partial I_1}-\frac{\rho^2}{r_0^2\lambda_1^2}\frac{\partial W}{\partial I_2}\right\}+p,\\ t_{\vartheta\vartheta}&=2\left\{\frac{\rho^2}{r_0^2}(l^2\lambda_2^2+m^2\lambda_3^2)\frac{\partial W}{\partial I_1}-\frac{r_0^2}{\rho^2}\left(\frac{l^2}{\lambda_2^2}+\frac{m^2}{\lambda_3^2}\right)\frac{\partial W}{\partial I_2}\right\}+p,\\ t_{\zeta\zeta}&=2\left\{(m^2\lambda_2^2+l^2\lambda_3^2)\frac{\partial W}{\partial I_1}-\left(\frac{m^2}{\lambda_2^2}+\frac{l^2}{\lambda_3^2}\right)\frac{\partial W}{\partial I_2}\right\}+p,\\ t_{\vartheta\zeta}&=2lm(\lambda_2^2-\lambda_3^2)\left\{\frac{\rho}{r_0}\frac{\partial W}{\partial I_1}+\frac{r_0\lambda_1^2}{\rho}\frac{\partial W}{\partial I_2}\right\},\\ t_{\zeta\rho}&=t_{\rho\vartheta}=0,\end{aligned}\right\} \tag{5·7}$$

the incompressibility condition (5·2) being satisfied by virtue of (4·5).

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Since I_1 and I_2 are functions of ρ alone, the stress components are independent of ϑ and ζ, and in the absence of body forces, the equations of equilibrium reduce to

$$\frac{\mathrm{d}t_{\rho\rho}}{\mathrm{d}\rho}+\frac{t_{\rho\rho}-t_{\vartheta\vartheta}}{\rho}=0, \tag{5·8}$$

the remaining equations being satisfied identically. Introducing the expressions (5·7) into (5·8), we obtain by integration

$$p=-2\left\{\frac{r_0^2\lambda_1^2}{\rho^2}\frac{\partial W}{\partial I_1}-\frac{\rho^2}{r_0^2\lambda_1^2}\frac{\partial W}{\partial I_2}\right\}+W(\rho)+\kappa, \tag{5·9}$$

where κ is an arbitrary constant, and we have employed (5·6) in the evaluation of the integral terms. The first three formulae of (5·7) now reduce to

$$t_{\rho\rho}=W(\rho)+\kappa,\quad t_{\vartheta\vartheta}=\frac{\mathrm{d}}{\mathrm{d}\rho}\{\rho W(\rho)\}+\kappa,\quad t_{\zeta\zeta}=F(\rho)+W(\rho)+\kappa, \tag{5·10}$$

where
$$F(\rho)=2\left\{\left[m^2\lambda_2^2+l^2\lambda_3^2-\frac{r_0^2\lambda_1^2}{\rho^2}\right]\frac{\partial W}{\partial I_1}-\left[\frac{m^2}{\lambda_2^2}+\frac{l^2}{\lambda_3^2}-\frac{\rho^2}{r_0^2\lambda_1^2}\right]\frac{\partial W}{\partial I_2}\right\}, \tag{5·11}$$

and equations (5·6) have again been used to simplify the expression for $t_{\vartheta\vartheta}$.

6. The surface tractions

The components of surface traction X_i acting at any point of the deformed surface are given by the relations

$$X_i=l'_j t_{ij}, \tag{6·1}$$

where X_1, X_2, X_3 act in the directions of r, θ, z respectively and are measured per unit area of deformed surface, l'_i are the direction cosines of the outward drawn normal at the point of the deformed surface under consideration, referred to this co-ordinate system, and t_{ij} are the values of the stress components defined by equations (5·4) at that point.

We shall consider the elastic material to be divided into two separate parts by the layer of cords, which may be regarded as a continuous thin sheet if the cords are sufficiently close together. We denote by R_1 and Π_1 the radial components of the surface tractions acting in the outward directions over the surfaces $\rho=r_1$, $\rho=r_0$ respectively on the elastic material contained between them; R_2, Π_2 are similarly defined for the surfaces $\rho=r_2$, $\rho=r_0$ respectively of the remainder of the material. These forces are all measured per unit area of deformed surface. The tangential components of surface traction on the cylindrical boundaries are evidently zero, since at each point we have $(l'_1, l'_2, l'_3)=(\pm 1, 0, 0)$, and from (5·7) $t_{\rho\vartheta}=t_{\zeta\rho}=0$. From (5·10) and (6·1) we have for the radial components

$$\left.\begin{aligned}R_1&=W(r_1)+\kappa_1,\quad \Pi_1=W(r_0)+\kappa_1,\\R_2&=W(r_2)+\kappa_2,\quad \Pi_2=W(r_0)+\kappa_2,\end{aligned}\right\} \tag{6·2}$$

the constant κ in (5·10) being given the different values κ_1 and κ_2 in the outer and inner parts of the elastic material respectively. From (6·2), the resultant force P acting radially outwards on the layer of cords is given by

$$P=\Pi_1-\Pi_2=R_1-R_2-W(r_1)+W(r_2), \tag{6·3}$$

and is evidently zero if $\kappa_1=\kappa_2$.

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From (4·9) we may obtain, by a straightforward calculation, the direction cosines at points of the surfaces of the deformed body which were originally on the planes $y = \pm b$, $z = \pm c$. The surface tractions at these points may then be found by combining (5·7) and (5·10) with (6·1), provided κ is given the appropriate values κ_1 for $r_1 \geqslant \rho \geqslant r_0$ and κ_2 for $r_0 \geqslant \rho \geqslant r_2$. This calculation is omitted since the resulting expressions are complicated in form and are not required subsequently.

7. The tensions in the cords

The deformation described by (4·3) subjects the layer of cords to a pure homogeneous strain of the type considered in §2, and the forces appropriate to this deformation may be calculated by formulae analogous to (2·7). After flexure, the deformed layer of cords forms part of a cylindrical surface in which, owing to the tensions in the cords, we may define stress resultants as follows:

T_1 and S are the forces per unit length, measured in the deformed state, acting in the longitudinal and azimuthal directions respectively on an element of length in the sheet which lies in the azimuthal direction;

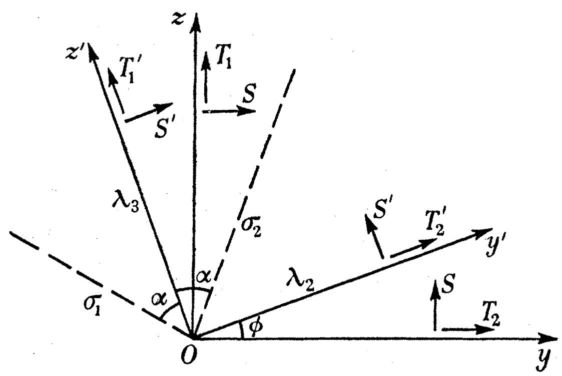

Figure 2

T_2 and S are the forces per unit length, measured in the deformed state, acting in the azimuthal and longitudinal directions respectively on an element of length which lies in the longitudinal direction. The equality in the values of S can readily be demonstrated.

The principal directions of strain in the layer of cords are, from §4, obtained from the directions of T_1, T_2 and S by rotation through an angle ϕ measured anticlockwise. Thus, if the stress resultants in these principal directions corresponding to T_1, T_2 and S are denoted by T_1', T_2', and S' respectively, we have, by the usual formulae for rotation of axes (see, for example, Love 1952, §49), and remembering the sense in which T_1, T_2 are defined

$$\left.\begin{aligned} T_1 &= l^2 T_1' + m^2 T_2' + 2lmS', \\ T_2 &= m^2 T_1' + l^2 T_2' - 2lmS', \\ S &= -lm(T_1' - T_2') + (l^2 - m^2) S', \end{aligned}\right\} \tag{7·1}$$

The orientation of the quantities defined for the layer of cords is illustrated in figure 2, the broken lines indicating the directions of the cords; the positive direction of the x-axis is

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considered to be outward from the paper and normal to it. T'_1, T'_2 and S' are therefore given by the formulae (2·7) for T''_1, T''_2 and S respectively, provided λ_1 is replaced by λ_3 and σ_1, σ_2 are interchanged. Thus

$$\left.\begin{aligned} T'_1 &= (\lambda_3/\lambda_2)\,(\sigma_1+\sigma_2)\cos^2\alpha,\\ T'_2 &= (\lambda_2/\lambda_3)\,(\sigma_1+\sigma_2)\sin^2\alpha,\\ S' &= -\tfrac{1}{2}(\sigma_1-\sigma_2)\sin 2\alpha, \end{aligned}\right\} \tag{7·2}$$

and introducing these relations into (7·1) we obtain

$$\left.\begin{aligned} T_1 &= \lambda_1\{(\lambda_3 l\cos\alpha-\lambda_2 m\sin\alpha)^2\sigma_1+(\lambda_3 l\cos\alpha+\lambda_2 m\sin\alpha)^2\sigma_2\},\\ T_2 &= \lambda_1\{(\lambda_3 m\cos\alpha+\lambda_2 l\sin\alpha)^2\sigma_1+(\lambda_3 m\cos\alpha-\lambda_2 l\sin\alpha)^2\sigma_2\},\\ S &= \{\lambda_1 lm(\lambda_2^2\sin^2\alpha-\lambda_3^2\cos^2\alpha)-\tfrac{1}{2}(l^2-m^2)\sin 2\alpha\}\,\sigma_1\\ &\quad+\{\lambda_1 lm(\lambda_2^2\sin^2\alpha-\lambda_3^2\cos^2\alpha)+\tfrac{1}{2}(l^2-m^2)\sin 2\alpha\}\,\sigma_2. \end{aligned}\right\} \tag{7·3}$$

For radial equilibrium of the layer of cords we must have

$$T_2 = Pr_0, \tag{7·4}$$

where P is given by (6·3). Combining (6·3) and (7·3) with (7·4) we thus obtain

$$\begin{aligned} \lambda_1\{(\lambda_3 m\cos\alpha+\lambda_2 l\sin\alpha)^2\sigma_1+(\lambda_3 m\cos\alpha-\lambda_2 l\sin\alpha)^2\sigma_2\}&\\ = r_0\{R_1-R_2-W(r_1)+W(r_2)\}.& \end{aligned} \tag{7·5}$$

It has been seen from §4 that when the dimensions of the cuboid and the angle ϕ which specifies the orientation of the cords are given, the deformation is completely determined by any one of the principal extension ratios λ_1, λ_2, λ_3 together with the radius of curvature which any given x-plane in the cuboid assumes during flexure. For convenience we may choose λ_3 and r_0 to determine the deformation. If the surface tractions R_1, R_2 are given on the cylindrical boundaries of the deformed body, κ_1 and κ_2 are given by (6·2), and then the stresses and the surface tractions on the remaining boundaries may be evaluated from (5·7), (5·10) and (6·1). Also, remembering the definition $(\sigma_1,\sigma_2)=(\tau_1/d_1,\tau_2/d_2)$, we see that a knowledge of the four quantities λ_3, r_0, R_1, R_2 is sufficient to determine, by means of (7·5), one relation between the tensions τ_1, τ_2 in the individual cords. A further relation may be obtained from (7·3) by assigning a given value to T_1 or S. If $\sigma_1=\sigma_2=\sigma$, the tensions in the cords are completely determined by (7·5). We have

$$\frac{\tau_1}{d_1}=\frac{\tau_2}{d_2}=\sigma=\frac{r_0\{R_1-R_2-W(r_1)+W(r_2)\}}{2\lambda_1\{\lambda_2^2 l^2\sin^2\alpha+\lambda_3^2 m^2\cos^2\alpha\}}, \tag{7·6}$$

and the formulae (7·3) then yield

$$\left.\begin{aligned} T_1 &= \frac{r_0(\lambda_2^2 m^2\sin^2\alpha+\lambda_3^2 l^2\cos^2\alpha)}{\lambda_2^2 l^2\sin^2\alpha+\lambda_3^2 m^2\cos^2\alpha}\{R_1-R_2-W(r_1)+W(r_2)\},\\ T_2 &= r_0\{R_1-R_2-W(r_1)+W(r_2)\},\\ S &= \frac{lmr_0(\lambda_2^2\sin^2\alpha-\lambda_3^2\cos^2\alpha)}{\lambda_2^2 l^2\sin^2\alpha+\lambda_3^2 m^2\cos^2\alpha}\{R_1-R_2-W(r_1)+W(r_2)\}. \end{aligned}\right\} \tag{7·7}$$

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If $R_1 = R_2 = 0$ and the cords are unstressed, we have from (7·6)

$$W(r_1) = W(r_2),$$

which is true if

$$I_1(r_1) = I_1(r_2), \quad I_2(r_1) = I_2(r_2),$$

and this, with (5·6), yields

$$\frac{r_0^4}{r_1^2 r_2^2} = \frac{l^2\lambda_2^2 + m^2\lambda_3^2}{\lambda_1^2} = \frac{1}{\lambda_1^2\left(\dfrac{l^2}{\lambda_2^2} + \dfrac{m^2}{\lambda_3^2}\right)}. \tag{7·8}$$

Unless $\phi = 0$ or $\frac{1}{2}\pi$, this condition can only be satisfied if $\lambda_2 = \lambda_3 = 1$, i.e. if the layer of cords is unstretched in the deformed cuboid. We then have $\lambda_1 = 1$ and $r_0^2 = r_1 r_2$.

8. The symmetrical case

When the cords are symmetrically placed with respect to the axis of flexure, so that $\phi = 0$, the results assume a much simpler form. Thus, remembering (4·4), we may put

$$l = 1, \quad m = 0, \quad K_1 = \lambda_2, \quad K_2 = \lambda_3, \quad L = 0, \tag{8·1}$$

in the formulae of §§ 4 to 7. In particular, from (4·9) we have

$$\rho = (2r_0\lambda_1 x + B)^{\frac{1}{2}}, \quad \vartheta = \lambda_2 y/r_0, \quad \zeta = \lambda_3 z, \tag{8·2}$$

and from (7·1), $(T_1, T_2, S) = (T_1', T_2', S')$, the latter quantities being given by (7·2). These relations when combined with (7·4) yield

$$\left.\begin{aligned}\sigma_1 + \sigma_2 &= \lambda_3 r_0 P/(\lambda_2 \sin^2\alpha),\\ T_1 &= (\lambda_3^2/\lambda_2^2)\, r_0 \cot^2\alpha P,\end{aligned}\right\} \tag{8·3}$$

where P is again given by (6·3).

From (8·2), planes normal to the y-axis in the unstrained body become, in the deformed state, planes containing the z-axis, and the planes z = constant in the unstrained body remain normal to the z-axis after deformation. The direction cosines l_i' defined in §6 therefore take the values $(0, \pm 1, 0)$, $(0, 0, \pm 1)$ respectively on these surfaces. Also, since from (5·7) and (8·1), $t_{\vartheta\zeta} = t_{\zeta\rho} = t_{\rho\vartheta} = 0$, the tangential components of the surface tractions are now zero on these planes. Denoting the normal components measured per unit area of deformed surface by Θ, Z at points on the planes ϑ = constant, ζ = constant respectively in the deformed body, we have, from (6·1) and (5·10),

$$\Theta = \frac{\mathrm{d}}{\mathrm{d}\rho}\{\rho W(\rho)\} + \kappa, \tag{8·4}$$

$$Z = G(\rho) + W(\rho) + \kappa, \tag{8·5}$$

where, from (5·11),

$$G(\rho) = [F(\rho)]_{l=1,\, m=0,} = 2\left(\lambda_3^2 - \frac{r_0^2\lambda_1^2}{\rho^2}\right)\left(\frac{\partial W}{\partial I_1} + \frac{\rho^2\lambda_2^2}{r_0^2}\frac{\partial W}{\partial I_2}\right), \tag{8·6}$$

and in (8·4) and (8·5) κ must be given the value appropriate to the section of the surface under consideration. Thus, from (6·2),

$$\left.\begin{aligned}\kappa = \kappa_1 &= R_1 - W(r_1) \quad \text{when} \quad r_1 \geqslant \rho \geqslant r_0,\\ \text{and} \quad \kappa = \kappa_2 &= R_2 - W(r_2) \quad \text{when} \quad r_0 \geqslant \rho \geqslant r_2.\end{aligned}\right\} \tag{8·7}$$

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If 2ψ is the angle subtended at the z-axis by the deformed cuboid we have, from (8·2), $\psi = \lambda_2 b/r_0$, and from (8·3), (8·5), (8·7) and (6·3), the resultant normal force F_1 acting on either of the faces which were initially at $z = \pm c$ is then given by

$$F_1 = 2\left\{r_0\psi T_1 + \int_{r_2}^{r_1} Z\psi\rho\,\mathrm{d}\rho\right\}$$
$$= \frac{\lambda_2 b}{r_0}\left\{2\int_{r_2}^{r_1}\rho[G(\rho)+W(\rho)]\,\mathrm{d}\rho + Pr_0^2\left[2\frac{\lambda_3^2}{\lambda_2^2}\cot^2\alpha - 1\right]\right.$$
$$\left. + [R_1 - W(r_1)]\,r_1^2 - [R_2 - W(r_2)]\,r_2^2\right\}. \tag{8·8}$$

Similarly, from (7·4), (6·3), (8·4) and (8·7), we may calculate the resultant normal force F_2 on each of the surfaces initially at $y = \pm b$. Thus

$$F_2 = 2\lambda_3 c\left\{T_2 + \int_{r_2}^{r_1}\Theta\,\mathrm{d}\rho\right\} = 2\lambda_3 c(R_1 r_1 - R_2 r_2). \tag{8·9}$$

If $R_1 = R_2 = 0$, F_2 is zero and the forces acting on these surfaces are statically equivalent to a couple M given by

$$M = 2\lambda_3 c\left\{T_2 r_0 + \int_{r_2}^{r_1}\Theta\rho\,\mathrm{d}\rho\right\} = \lambda_3 c\int_{r_2}^{r_1}(\rho^2 - r_0^2)\frac{\mathrm{d}W}{\mathrm{d}\rho}\,\mathrm{d}\rho. \tag{8·10}$$

If $R_1 = R_2 = 0$, and in addition $\sigma_1 = -\sigma_2 = \sigma$, we have, from (7·3), (7·4) and (8·1),

$$T_1 = T_2 = 0, \quad S = -\sigma\sin 2\alpha, \quad P = 0. \tag{8·11}$$

Then from (6·3)
$$W(r_1) = W(r_2) = W_0 \quad \text{(say)}, \tag{8·12}$$

and equations (8·8) and (8·10) yield

$$\left.\begin{aligned} F_1 &= \frac{\lambda_2 b}{r_0}\left\{2\int_{r_2}^{r_1}\rho[G(\rho)+W(\rho)]\,\mathrm{d}\rho - (r_1^2 - r_2^2)\,W_0\right\}, \\ M &= \lambda_3 c\int_{r_2}^{r_1}\rho^2\frac{\mathrm{d}W}{\mathrm{d}\rho}\,\mathrm{d}\rho. \end{aligned}\right\} \tag{8·13}$$

Also, the conditions (8·12) are true if

$$I_1(r_1) = I_1(r_2), \quad I_2(r_1) = I_2(r_2), \tag{8·14}$$

and from (5·6) and (8·1) we then have

$$r_0 = (\lambda_2 r_1 r_2/\lambda_1)^{\frac{1}{2}}, \tag{8·15}$$

which, with (2·4), (4·6) and (4·10), yields a relationship between r_0 and λ_1.

9. The simple extension and flexure of a thin sheet

By allowing the x-dimension of the cuboid of §8 to become small compared with the radius of flexure, we may examine, by a limiting process, the bending of a thin reinforced sheet containing cords symmetrically placed with respect to the axis of flexure. Thus, if we write

$$\eta_i = a_i - a_0, \quad \epsilon_i = r_i - r_0 \quad (i = 1, 2), \tag{9·1}$$

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where η_i, ϵ_i are small compared with r_0, the undeformed body becomes a sheet of thickness $\eta_1 - \eta_2$, bounded in a yz-plane by the lines $y = \pm b$, $z = \pm c$. We shall restrict our attention to the case where the major surfaces of the sheet are free from applied forces so that $R_1 = R_2 = 0$, and then from (8·9), $F_2 = 0$. Also, we shall suppose $S = 0$, so that from (7·2) we may put $\sigma_1 = \sigma_2 = \sigma$. An approximate expression for the flexural couple M may now be obtained by expanding (8·10) as a power series in ϵ_1, ϵ_2, and observing that for any function $f(\rho)$, if $\rho = r_0 + \epsilon_0$, we have

$$\int_{r_2}^{r_1} f(\rho)\,\mathrm{d}\rho = \int_{\epsilon_2}^{\epsilon_1} f(r_0+\epsilon)\,\mathrm{d}\epsilon = (\epsilon_1-\epsilon_2) f(r_0) + \tfrac{1}{2}(\epsilon_1^2-\epsilon_2^2)\, f'(r_0) + \tfrac{1}{6}(\epsilon_1^3-\epsilon_2^3)\, f''(r_0) + \ldots. \quad (9\cdot2)$$

This process yields

$$M = \lambda_3 c\left\{r_0(\epsilon_1^2-\epsilon_2^2)\left[\frac{\mathrm{d}W}{\mathrm{d}\rho}\right]_{\rho=r_0} + \tfrac{1}{3}(\epsilon_1^3-\epsilon_2^3)\left(\left[\frac{\mathrm{d}W}{\mathrm{d}\rho}\right]_{\rho=r_0} + 2r_0\left[\frac{\mathrm{d}^2W}{\mathrm{d}\rho^2}\right]_{\rho=r_0}\right) + \ldots\right\}. \quad (9\cdot3)$$

Also from (4·10) and (9·1) we have

$$\frac{\epsilon_i}{r_0} = \left(1+\frac{2\lambda_1\eta_i}{r_0}\right)^{\frac{1}{2}} - 1 = \frac{\lambda_1\eta_i}{r_0}\left(1 - \frac{\lambda_1\eta_i}{2r_0} + \frac{\lambda_1^2\eta_i^2}{2r_0^2} - \ldots\right), \quad (9\cdot4)$$

and from (5·6) with $l = 1$, $m = 0$

$$\frac{\mathrm{d}I_1}{\mathrm{d}\rho} = \frac{1}{\lambda_3^2}\frac{\mathrm{d}I_2}{\mathrm{d}\rho} = -2\left(\frac{r_0^2\lambda_1^2}{\rho^3} - \frac{\rho\lambda_2^2}{r_0^2}\right). \quad (9\cdot5)$$

Introducing these expressions into (9·3) we obtain

$$M = \frac{2\lambda_1 c r_0^2}{\lambda_2}\left\{(\lambda_2^2-\lambda_1^2)\frac{\eta_1^2-\eta_2^2}{r_0^2}DW + \frac{4\lambda_1}{3}\frac{\eta_1^3-\eta_2^3}{r_0^3}[2\lambda_1^2 DW + (\lambda_2^2-\lambda_1^2)^2 D^2W] + O\left(\frac{\eta^4}{r_0^4}\right)\right\} \quad (\eta = \rho - r_0), \quad (9\cdot6)$$

where $D \equiv \partial/\partial I_1 + \lambda_3^2\,\partial/\partial I_2$ and the derivatives of W with respect to I_1 and I_2 are evaluated at $\rho = r_0$ (or $\eta = 0$). By similar procedure we may obtain from (8·3) and (8·8) expressions for σ and F_1 which agree, to a first approximation, with the results obtained for simple extension in § 3.

From (9·6) it is evident that the couple required to maintain a given state of flexure depends upon the form of strain-energy function for the elastic material of the sheet, in addition to the disposition of the cords. For rubber-like materials $\partial W/\partial I_1$ and $\partial W/\partial I_2$ are positive, and the work of Rivlin & Saunders (1951) suggests that for a natural rubber vulcanizate these quantities are much larger numerically than the higher order derivatives of W with respect to the invariants. In this case, since

$$\mathrm{d}/\mathrm{d}\rho = (\mathrm{d}I_1/\mathrm{d}\rho)\, D \equiv (\mathrm{d}I_1/\mathrm{d}\rho)\, (\partial/\partial I_1 + \lambda_3^2\,\partial/\partial I_2),$$

it follows that the terms involving these higher order derivatives can only become important in any term of the expansion if λ_3 is large. Since λ_3 is restricted by the relation (4·6) to be appreciably less than $\sec^2\alpha$, we shall suppose that it is sufficiently small for these higher-order terms not to assume overriding importance.

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If the cords lie in a plane midway between the two major surfaces of the undeformed sheet we may put $\eta_1 = -\eta_2 = \eta_0$ in (9·6). If r_0 is sufficiently large, the value of M is approximately equal to that of the second term in this expansion, and we then have

$$M = \frac{16\lambda_1^2 c\eta_0^3}{3\lambda_2 r_0}\{2\lambda_1^2 DW + (\lambda_2^2-\lambda_1^2)^2 D^2W\}. \tag{9·7}$$

To this order of approximation the couple required to produce flexure is directly proportional to the curvature of the deformed sheet. If r_0 is not large compared with η_0, higher order terms may become important in the expansion for M, and the exact expression (8·10) must then be employed.

If the layer of cords does not lie exactly midway between the major surfaces of the undeformed sheet, $\eta_1+\eta_2 \neq 0$, and for sufficiently large values of r_0 the flexural couple is given by the first term of the expansion (9·6). Since this term is independent of the curvature of the deformed sheet we may infer that the sheet will bend under the application of a simple tensile force unless a suitable couple is also applied with its axis parallel to the direction of the force, the magnitude and direction of this couple depending upon the magnitude of the force and the disposition of the cords. The sign of the first term in the expansion (9·6) for M indicates the sense in which this couple must be applied. This is dependent upon the signs of the factors $\eta_1+\eta_2$ and $\lambda_2^2-\lambda_1^2$, since DW and $\eta_1-\eta_2$ are both positive. By analysis similar to that of § 3 it may be shown that $\lambda_2^2-\lambda_1^2$ is positive if $\frac{1}{2}\pi > \alpha > \cos^{-1}(\lambda_3^2+\lambda_3+1)^{-\frac{1}{2}}$ and negative if $\cos^{-1}(\lambda_3^2+\lambda_3+1)^{-\frac{1}{2}} > \alpha > \cos^{-1}(1/\lambda_3)$. Also, as in § 3, the sign of σ is opposite to that of $\lambda_2^2-\lambda_1^2$. We thus have

$$\left.\begin{array}{llll} M>0, & \sigma<0 & \text{if} \quad \eta_1+\eta_2>0 & \text{and} \quad \frac{1}{2}\pi>\alpha>\cos^{-1}(\lambda_3^2+\lambda_3+1)^{-\frac{1}{2}}, \\ M>0, & \sigma>0 & \text{if} \quad \eta_1+\eta_2<0 & \text{and} \quad \cos^{-1}(\lambda_3^2+\lambda_3+1)^{-\frac{1}{2}}>\alpha>\cos^{-1}(1/\lambda_3), \\ M<0, & \sigma>0 & \text{if} \quad \eta_1+\eta_2>0 & \text{and} \quad \cos^{-1}(\lambda_3^2+\lambda_3+1)^{-\frac{1}{2}}>\alpha>\cos^{-1}(1/\lambda_3), \\ M<0, & \sigma<0 & \text{if} \quad \eta_1+\eta_2<0 & \text{and} \quad \frac{1}{2}\pi>\alpha>\cos^{-1}(\lambda_3^2+\lambda_3+1)^{-\frac{1}{2}}. \end{array}\right\} \tag{9·8}$$

Hitherto we have assumed that $r_1 \geq r_0 \geq r_2 > 0$, $a_1 \geq a_0 \geq a_2 > 0$, so that, from (9·1), if $\eta_1+\eta_2>0$ the plane containing the cords in the undeformed sheet lies nearer to the major surface which becomes concave after flexure; similarly, if $\eta_1+\eta_2<0$ the cords lie nearer to the opposite boundary surface. Moreover, from (9·7) the expression for the couple required to produce this flexure is positive when the cords lie midway between the major surfaces of the sheet. If, therefore, the first term of (9·6) is positive, a positive couple must be applied to prevent bending, and on removal of this couple, flexure would occur in a sense opposite to that previously assumed. Hence the application of a simple tensile force alone will produce flexure in such a sense that the major surface of the undeformed sheet which lies nearer to the plane of the cords becomes concave after deformation if

$$\cos^{-1}(\lambda_3^2+\lambda_3+1)^{-\frac{1}{2}} > \alpha > \cos^{-1}(1/\lambda_3),$$

and convex if

$$\tfrac{1}{2}\pi > \alpha > \cos^{-1}(\lambda_3^2+\lambda_3+1)^{-\frac{1}{2}}.$$

If $\cos^{-1}(\lambda_3^2+\lambda_3+1)^{-\frac{1}{2}} > \alpha > \cos^{-1}(1/\sqrt{3})$ the first term of the expansion (9·6) changes sign as the extension ratio is increased from unity to its final value λ_3, and we may then expect that if the extending force is applied slowly, the direction of flexure will reverse during deformation.

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If M becomes zero for a sufficiently large value of r_0, the radius of flexure of the layer of cords in the deformed sheet may be obtained from the first and second terms of (9·6). Thus we have, approximately, $M = 0$ if

$$r_0 = -\frac{4\lambda_1(\eta_1^2+\eta_1\eta_2+\eta_2^2)\{2\lambda_1^2 DW+(\lambda_2^2-\lambda_1^2)^2 D^2W\}}{3(\eta_1+\eta_2)(\lambda_2^2-\lambda_1^2) DW}. \tag{9·9}$$

This yields a large value of r_0 if $(\eta_1^2+\eta_1\eta_2+\eta_2^2)/(\eta_1+\eta_2)$ is large compared with η_1 and η_2, i.e. if the denominator of this fraction is small compared with the numerator. The plane of the cords then lies near to the middle plane of the undeformed sheet.

If the elastic material has the Mooney form of strain-energy function so that

$$W = C_1(I_1-3)+C_2(I_2-3), \tag{9·10}$$

where C_1 and C_2 are constants, the expression (9·9) for the radius of flexure reduces to

$$r_0 = -\frac{8\lambda_1^3(\eta_1^2+\eta_1\eta_2+\eta_2^2)}{3(\lambda_2^2-\lambda_1^2)(\eta_1+\eta_2)}. \tag{9·11}$$

SIMULTANEOUS EXTENSION, INFLATION AND TORSION OF A REINFORCED CYLINDRICAL TUBE

10. Preliminary relations

The methods of the preceding sections may be employed to examine the simultaneous extension, inflation and torsion of a cylindrical tube of isotropic incompressible material which is reinforced by means of layers of inextensible cords lying in cylindrical surfaces co-axial with its curved boundaries. It will first be convenient, however, to quote the corresponding results for an unreinforced tube, obtained by Rivlin (1949 *b*), in a form suitable for subsequent applications.

We assume the undeformed tube to be bounded in the cylindrical polar co-ordinate system (r, θ, z) by the surfaces $r = a_1$, $r = a_2$ $(a_1 > a_2)$, $z = \pm l$, and suppose the resultant deformation to be produced by three successive deformations as follows:

(i) a uniform simple extension of ratio λ;

(ii) a uniform inflation in which the length of the tube remains constant and its external and internal radii change to $\mu_1 a_1$ and $\mu_2 a_2$ respectively;

(iii) a uniform simple torsion in which the angle of twist is ψ per unit length of the extended tube. A point initially at (r, θ, z) is therefore displaced by the resultant deformation to (ρ, ϑ, ζ), where

$$(\rho, \vartheta, \zeta) = (\mu r, \theta+\psi\lambda z, \lambda z), \tag{10·1}$$

μ being a function of r which assumes the values μ_1, μ_2 on the outer and inner surfaces of the tube respectively.

The formulae (5·1) to (5·4) and (5·8) again apply. Thus using (10·1), the incompressibility condition becomes

$$K = \lambda\rho^2 - r^2 = r^2(\lambda\mu^2-1), \tag{10·2}$$

where K is an arbitrary constant. The strain invariants are given by

$$\left.\begin{aligned} I_1 &= \lambda^2+\mu^2+\frac{1}{\lambda^2\mu^2}+\psi^2\lambda^2\mu^2r^2, \\ I_2 &= \frac{1}{\lambda^2}+\frac{1}{\mu^2}+\lambda^2\mu^2+\psi^2r^2, \end{aligned}\right\} \tag{10·3}$$

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and the stress components may be obtained from (5·4) and (5·8) in the forms

$$\left.\begin{aligned} t_{\rho\rho} &= -\int_{a_1}^{r} f(r)\,\mathrm{d}r + \kappa,\\ t_{\vartheta\vartheta} &= -\int_{a_1}^{r} f(r)\,\mathrm{d}r - \lambda\mu^2 r f(r) + \kappa,\\ t_{\zeta\zeta} &= -\int_{a_1}^{r} f(r)\,\mathrm{d}r + g(r) + \kappa,\\ t_{\vartheta\zeta} &= 2\psi r\left\{\lambda^2\mu\frac{\partial W}{\partial I_1} + \frac{1}{\mu}\frac{\partial W}{\partial I_2}\right\},\\ t_{\zeta\rho} &= t_{\rho\vartheta} = 0, \end{aligned}\right\} \tag{10·4}$$

where

$$\left.\begin{aligned} f(r) &= 2\left\{\left[\frac{1}{\lambda^2\mu^2} - \mu^2(1+\psi^2\lambda^2 r^2)\right]\frac{\partial W}{\partial I_1} - \left(\lambda^2\mu^2 - \frac{1}{\mu^2}\right)\frac{\partial W}{\partial I_2}\right\}\Big/(\lambda\mu^2 r),\\ g(r) &= 2\left\{\left(\lambda^2 - \frac{1}{\lambda^2\mu^2}\right)\frac{\partial W}{\partial I_1} + \left(\lambda^2\mu^2 - \frac{1}{\lambda^2} - \psi^2 r^2\right)\frac{\partial W}{\partial I_2}\right\}, \end{aligned}\right\} \tag{10·5}$$

and κ is a constant of integration.

The surface tractions may be obtained by combining the expressions (10·4) for the stress components with (6·1), and making use of (10·1) to determine l'_i. It is readily seen that the tangential components are zero on the curved surfaces initially at $r = a_1$, $r = a_2$, and if we denote the radial components directed outwards, and measured per unit area of deformed surface, by R_1, R_2 respectively, we have, since $(l'_1, l'_2, l'_3) = (\pm 1, 0, 0)$ on these surfaces,

$$\kappa = R_1 = R_2 + \int_{a_1}^{a_2} f(r)\,\mathrm{d}r. \tag{10·6}$$

Similarly, if $(R_\nu, \Theta_\nu, Z_\nu)$ are the components of surface traction in the radial, azimuthal and longitudinal directions respectively acting on the plane ends initially at $z = \pm l$, and again measured per unit area of deformed surface, we have, from (10·4), (10·1) and (6·1),

$$\left.\begin{aligned} &R_\nu = 0, \quad \Theta_\nu = 2\psi\lambda r\left\{\lambda\mu\frac{\partial W}{\partial I_1} + \frac{1}{\lambda\mu}\frac{\partial W}{\partial I_2}\right\},\\ &Z_\nu = -\int_{a_1}^{r} f(r)\,\mathrm{d}r + g(r) + \kappa. \end{aligned}\right\} \tag{10·7}$$

The resultant couple M and longitudinal force F on the plane ends of the tube are now given by

$$M = 2\pi\int_{\mu_2 a_2}^{\mu_1 a_1} \Theta_\nu \rho^2\,\mathrm{d}\rho = 4\pi\psi\int_{a_2}^{a_1} \mu r^3\left(\lambda\mu\frac{\partial W}{\partial I_1} + \frac{1}{\lambda\mu}\frac{\partial W}{\partial I_2}\right)\mathrm{d}r, \tag{10·8}$$

and

$$\left.\begin{aligned} N &= 2\pi\int_{\mu_2 a_2}^{\mu_1 a_1} Z_\nu \rho\,\mathrm{d}\rho\\ &= \frac{\pi}{\lambda}\left\{\int_{a_2}^{a_1}[2rg(r) + (r^2 - a_2^2)f(r)]\,\mathrm{d}r + (a_1^2 - a_2^2)\,R_1\right\},\\ &= \frac{\pi}{\lambda}\left\{\int_{a_2}^{a_1}[2rg(r) + (r^2 - a_1^2)f(r)]\,\mathrm{d}r + (a_1^2 - a_2^2)\,R_2\right\}, \end{aligned}\right\} \tag{10·9}$$

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if we make use of (10·1), (10·2) and (10·6), and employ the relation

$$\int_{a_2}^{a_1} r^2 f(r)\,\mathrm{d}r = a_2^2\int_{a_2}^{a_1} f(r)\,\mathrm{d}r - 2\int_{a_2}^{a_1} r\left\{\int_{a_1}^{r} f(r)\,\mathrm{d}r\right\}\mathrm{d}r$$

to simplify the formulae for N.

11. The relations for a layer of cords

We now consider the tube of the preceding section to be reinforced with a layer of thin inextensible cords, the path of each cord in the undeformed cylinder being a member of the family of similar circular helices

$$r = b, \quad z = b\theta\cot\alpha + \text{constant} \quad (a_1 \geqslant b \geqslant a_2), \tag{11·1}$$

where b is a constant. Each helix therefore cuts the generators of the cylinder $r = b$ in the undeformed tube at a constant angle α, and we shall suppose that when the tube undergoes the deformation described by (10·1) the cords make an angle β with the longitudinal direction. Also, we assume that before deformation the cords are spaced a distance d apart, this distance being measured along a path orthogonal to the helices (11·1), and that after deformation each cord carries a tension τ.

The restriction upon the deformation (10·1) imposed by the inextensible cords is readily obtained from geometrical considerations. Thus, if $\mathrm{d}s$ is an element of length lying along a path taken by one of the cords we have before deformation

$$\mathrm{d}s\cos\alpha = \mathrm{d}z, \quad \mathrm{d}s\sin\alpha = b\,\mathrm{d}\theta,$$

and after deformation

$$\mathrm{d}s\cos\beta = \mathrm{d}\zeta = \lambda\,\mathrm{d}z, \quad \mathrm{d}s\sin\beta = \mu_b b\,\mathrm{d}\vartheta = \mu_b b(\mathrm{d}\theta + \psi\lambda\,\mathrm{d}z),$$

where μ_b is the value of μ for $r = b$. Combining these relations we obtain

$$\cos\beta = \lambda\cos\alpha, \quad \sin\beta = \mu_b(\sin\alpha + \psi\lambda b\cos\alpha), \tag{11·2}$$

$$\mu_b^2 = \frac{1-\lambda^2\cos^2\alpha}{(\sin\alpha + \psi\lambda b\cos\alpha)^2}, \tag{11·3}$$

and from (10·2) and (11·3) we now have

$$K = b^2\left\{\frac{\lambda(1-\lambda^2\cos^2\alpha)}{(\sin\alpha+\psi\lambda b\cos\alpha)^2} - 1\right\}. \tag{11·4}$$

The layer of cords thus reduces by unity the number of degrees of freedom available to the deformation (10·1). Three such layers would yield three relationships of the form (11·4) between the constants λ, ψ and K which are employed to specify this deformation and, in principle, it would then be possible to determine explicitly the values of these constants. In this case a continuously varying deformation of the type (10·1) would not, in general, be possible, and we shall therefore, in subsequent work, restrict attention to tubes in which there are not more than two reinforcing sets of cords.

If the angle of the helices is unchanged by the deformation, so that $\alpha = \beta$, equations (11·2) yield

$$\lambda = 1, \quad \mu_b = 1/(1+\psi b\cot\alpha). \tag{11·5}$$

Stress resultants T_1, T_2, S in the deformed layer of cords may be defined exactly as in § 7. If n_ϑ, n_ζ denote respectively the number of cords which are intersected by unit length of a line of latitude and a generator in the deformed layer, we have, from geometrical considerations,

$$n_\vartheta = \cos\alpha/(\mu_b d), \quad n_\zeta = n_\vartheta\tan\beta.$$

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Remembering (11·2), we obtain, by a simple resolution of forces

$$\left.\begin{aligned} T_1 &= \lambda\cos^2\alpha\sigma/\mu_b, \\ T_2 &= \mu_b(\sin\alpha+\psi\lambda b\cos\alpha)^2\sigma/\lambda, \\ S &= \cos\alpha(\sin\alpha+\psi\lambda b\cos\alpha)\,\sigma. \end{aligned}\right\} \tag{11·6}$$

12. Simultaneous extension, inflation and torsion of a tube reinforced with a single set of cords

When the tube is reinforced by a single set of cords as described in the previous section, we may follow the procedure of § 6 and consider the elastic material to be divided into two separate parts by this layer, the difference in surface tractions on the neighbouring surfaces of these two parts being balanced by the tensions in the cords. We therefore denote by Π_1, Π_2 the normal components of the surface traction on the surfaces of the outer and inner parts of the elastic material adjacent to the layer of cords which is initially at $r = b$, these components being directed outwards from the elastic material on which they act and measured per unit area of deformed surface. From (6·1), (10·1) and (10·4) it follows that the tangential components on these surfaces are zero.

Application of (10·6) to the outer and inner cylinders of elastic material yields

$$\left.\begin{aligned} \kappa_1 &= R_1 = \Pi_1+\int_{a_1}^{b} f(r)\,\mathrm{d}r, \\ \text{and}\qquad \kappa_2 &= \Pi_2 = R_2+\int_{b}^{a_2} f(r)\,\mathrm{d}r, \end{aligned}\right\} \tag{12·1}$$

respectively, where κ_1, κ_2 are the appropriate values of κ for the parts of the material under consideration.

The stresses in the elastic material and the surface tractions on the plane ends may now be obtained from (10·4) and (10·7) respectively by writing κ_1 for κ if $a_1 \geqslant r \geqslant b$ and κ_2, b for κ, a_1 if $b \geqslant r \geqslant a_2$, κ_1 and κ_2 being given by (12·1). The resultant pressure acting radially outwards on the layer of cords is given by

$$P = \Pi_1-\Pi_2 = R_1-R_2-\int_{a_1}^{a_2} f(r)\,\mathrm{d}r, \tag{12·2}$$

and this is connected with the stress resultant T_2 by the relation

$$T_2 = P\mu_b b. \tag{12·3}$$

From (11·6), (12·2) and (12·3) we now have

$$\sigma = \frac{\lambda b\left\{R_1-R_2-\int_{a_1}^{a_2} f(r)\,\mathrm{d}r\right\}}{(\sin\alpha+\psi\lambda b\cos\alpha)^2}, \tag{12·4}$$

and

$$\left.\begin{aligned} T_1 &= \frac{\lambda^2 b\cos^2\alpha\left\{R_1-R_2-\int_{a_1}^{a_2} f(r)\,\mathrm{d}r\right\}}{\mu_b(\sin\alpha+\psi\lambda b\cos\alpha)^2}, \\ S &= \frac{\lambda b\cos\alpha\left\{R_1-R_2-\int_{a_1}^{a_2} f(r)\,\mathrm{d}r\right\}}{\sin\alpha+\psi\lambda b\cos\alpha}. \end{aligned}\right\} \tag{12·5}$$

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The resultant couple M acting on either of the plane ends of the composite tube initially at $z = \pm l$ may be evaluated by adding to the expression (10·8) for the couple on the elastic material that due to the layer of cords. Since this latter component is of magnitude $2\pi\mu_b^2 b^2 S$, employing (11·3) and (12·5), we obtain

$$M = 2\pi\left\{2\psi\int_{a_2}^{a_1}\mu r^3\left(\lambda\mu\frac{\partial W}{\partial I_1}+\frac{1}{\lambda\mu}\frac{\partial W}{\partial I_2}\right)\mathrm{d}r+\frac{\lambda b^3\cos\alpha(1-\lambda^2\cos^2\alpha)}{(\sin\alpha+\psi\lambda b\cos\alpha)^3}\left[R_1-R_2+\int_{a_2}^{a_1}f(r)\,\mathrm{d}r\right]\right\}. \quad (12·6)$$

The resultant longitudinal force on a plane end may be similarly obtained from (10·9) and (12·5). Thus if N_1 and N_2 denote the resultant forces on the plane ends of the cylinders of elastic material for which $a_1 \geqslant r \geqslant b$ and $b \geqslant r \geqslant a_2$ respectively, we have, from (10·9),

$$\left.\begin{aligned}N_1 &= \frac{\pi}{\lambda}\left\{\int_b^{a_1}[2rg(r)+(r^2-b^2)f(r)]\,\mathrm{d}r+(a_1^2-b^2)\,R_1\right\},\\ N_2 &= \frac{\pi}{\lambda}\left\{\int_{a_2}^{b}[2rg(r)+(r^2-b^2)f(r)]\,\mathrm{d}r+(b^2-a_2^2)\,R_2\right\},\end{aligned}\right\} \quad (12·7)$$

and employing (11·3) and (12·5) we obtain

$$\begin{aligned}N &= N_1+N_2+2\pi\mu_b bT_1\\ &= \frac{\pi}{\lambda}\left\{\int_{a_2}^{a_1}[2rg(r)+(r^2+\chi b^2)f(r)]\,\mathrm{d}r+R_1(a_1^2+\chi b^2)-R_2(a_2^2+\chi b^2)\right\},\end{aligned} \quad (12·8)$$

where

$$\chi = \frac{2\lambda^3\cos^2\alpha}{(\sin\alpha+\psi\lambda b\cos\alpha)^2}-1. \quad (12·9)$$

If the applied forces R_1, R_2, M and N are given, (12·6) and (12·8) may be regarded as two equations for the determination of ψ and λ.

If $\psi = -\tan\alpha/(\lambda b)$, since μ_b must remain finite, we have, from (11·2),

$$\beta = 0, \quad \lambda = \sec\alpha, \quad \psi = -\sin\alpha/b. \quad (12·10)$$

The cords then coincide with generators of the cylindrical surface in which they lie in the deformed tube, which thus attains its maximum extension. Also, for N to remain finite we have, from (12·8) and (12·9),

$$P = R_1-R_2+\int_{a_2}^{a_1}f(r)\,\mathrm{d}r = 0. \quad (12·11)$$

From (11·6) it is evident that $T_2 = S = 0$ and that σ and T_1 may have any arbitrary finite values which satisfy the first of (11·6), and from (12·8) N is also arbitrary to this extent. These conclusions are obvious from physical considerations.

13. Simultaneous extension, inflation and torsion of a tube reinforced with two sets of cords

The results of the preceding section may be readily extended to the case where the cylindrical tube described in § 10 is reinforced with two sets of cords. We suppose the cords to lie in the cylindrical surfaces $r = b$, $r = c$ in the undeformed body, where $a_1 \geqslant b \geqslant c \geqslant a_2$, and employ the notation of §§ 10 to 12, with the addition of suffixes b or c to distinguish quantities appropriate to either set of cords. The paths of the cords in the undeformed body may therefore be represented by the two families of helices

$$\left.\begin{aligned}r &= b, \quad z = b\theta\cot\alpha_b+\text{constant},\\ r &= c, \quad z = c\theta\cot\alpha_c+\text{constant}.\end{aligned}\right\} \quad (13·1)$$

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The incompressibility condition (10·2) now yields

$$\left.\begin{aligned} K = r^2(\lambda\mu^2-1) &= a_1^2(\lambda\mu_1^2-1) = a_2^2(\lambda\mu_2^2-1) \\ &= b^2(\lambda\mu_b^2-1) = c^2(\lambda\mu_c^2-1), \end{aligned}\right\} \tag{13·2}$$

and corresponding to (11·3) we have

$$\mu_b^2 = \frac{1-\lambda^2\cos^2\alpha_b}{(\sin\alpha_b+\psi\lambda b\cos\alpha_b)^2}, \quad \mu_c^2 = \frac{1-\lambda^2\cos^2\alpha_c}{(\sin\alpha_c+\psi\lambda c\cos\alpha_c)^2}. \tag{13·3}$$

From (13·2) and (13·3) we obtain

$$b^2\left\{\frac{\lambda(1-\lambda^2\cos^2\alpha_b)}{(\sin\alpha_b+\psi\lambda b\cos\alpha_b)^2}-1\right\} = c^2\left\{\frac{\lambda(1-\lambda^2\cos^2\alpha_c)}{(\sin\alpha_c+\psi\lambda c\cos\alpha_c)^2}-1\right\}. \tag{13·4}$$

For any given value of λ, equation (13·4) may be regarded as a biquadratic for the determination of ψ, and conversely, if ψ is known, (13·4) becomes a biquadratic in λ. The values of μ_b and μ_c obtained from (13·3) must evidently be positive for a physically real deformation.

Corresponding to the relations (12·1) we now have

$$\left.\begin{aligned} \kappa_1 &= R_1 = \Pi_{1b}+\int_{a_1}^{b} f(r)\,\mathrm{d}r, \\ \kappa_2 &= \Pi_{2b} = \Pi_{1c}+\int_{b}^{c} f(r)\,\mathrm{d}r, \\ \kappa_3 &= \Pi_{2c} = R_2+\int_{c}^{a_2} f(r)\,\mathrm{d}r, \end{aligned}\right\} \tag{13·5}$$

where κ_1, κ_2, κ_3 are the values of κ appropriate to the regions $a_1 \geqslant r \geqslant b$, $b \geqslant r \geqslant c$, $c \geqslant r \geqslant a_2$ respectively, and (Π_{1b}, Π_{2b}), (Π_{1c}, Π_{2c}) are the surface tractions acting on the surfaces of the elastic material adjacent to the layers of cords at $r = b$, $r = c$, and are defined in a manner exactly analogous to the quantities (Π_1, Π_2) of § 12. These surface tractions are related to the stress resultants in the layers of cords by relations of the form (12·3). Thus

$$T_{2b} = (\Pi_{1b}-\Pi_{2b})\,\mu_b b, \quad T_{2c} = (\Pi_{1c}-\Pi_{2c})\,\mu_c c. \tag{13·6}$$

From (13·5) and (13·6) we may obtain

$$\Pi_{2b} = R_1-\int_{a_1}^{b} f(r)\,\mathrm{d}r-\frac{T_{2b}}{\mu_b b}, \quad \Pi_{1c} = R_1-\int_{a_1}^{c} f(r)\,\mathrm{d}r-\frac{T_{2b}}{\mu_b b}, \tag{13·7}$$

and

$$\frac{T_{2b}}{\mu_b b}+\frac{T_{2c}}{\mu_c c} = R_1-R_2-\int_{a_1}^{a_2} f(r)\,\mathrm{d}r. \tag{13·8}$$

Also since formulae of the type (11·6) apply for each set of cords, we have, from (13·8),

$$\sigma_b(\sin\alpha_b+\psi\lambda b\cos\alpha_b)^2/b+\sigma_c(\sin\alpha_c+\psi\lambda c\cos\alpha_c)^2/c = \lambda\left\{R_1-R_2-\int_{a_1}^{a_2} f(r)\,\mathrm{d}r\right\}, \tag{13·9}$$

which corresponds to the relation (12·4).

The resultant couple M on a plane end of the tube may now be evaluated by combining the components due to each layer of cords with that arising from the deformation of the elastic material. The latter component is given by (10·8); the former components contribute a couple of magnitude $2\pi(\mu_b^2 b^2 S_b+\mu_c^2 c^2 S_c)$ where S_b and S_c are given by formulae analogous to the last of (11·6). Thus employing (13·3) we obtain

$$M = 2\pi\left\{2\psi\int_{a_2}^{a_1}\mu r^3\left(\lambda\mu\frac{\partial W}{\partial I_1}+\frac{1}{\lambda\mu}\frac{\partial W}{\partial I_2}\right)\mathrm{d}r+\frac{b^2\cos\alpha_b(1-\lambda^2\cos^2\alpha_b)\,\sigma_b}{\sin\alpha_b+\psi\lambda b\cos\alpha_b}+\frac{c^2\cos\alpha_c(1-\lambda^2\cos^2\alpha_c)\,\sigma_c}{\sin\alpha_c+\psi\lambda c\cos\alpha_c}\right\}. \tag{13·10}$$

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The resultant longitudinal force N on a plane end of the tube may be derived by a procedure similar to that employed in obtaining (12·8). Applying formulae analogous to (10·9) and (11·6) to each section of the tube, and making use of (13·5) to (13·8) we may obtain the alternative expressions

$$\left.\begin{aligned} N &= \frac{\pi}{\lambda}\Big\{\int_{a_2}^{a_1}[2rg(r)+(r^2-c^2)f(r)]\,\mathrm{d}r+(a_1^2-c^2)\,R_1+(c^2-a_2^2)\,R_2 \\ &\qquad -(b^2-c^2)\,(\sin\alpha_b+\psi\lambda b\cos\alpha_b)^2\,\sigma_b/(\lambda b)+2\lambda^2(\sigma_b b\cos^2\alpha_b+\sigma_c c\cos^2\alpha_c)\Big\}, \\ &= \frac{\pi}{\lambda}\Big\{\int_{a_2}^{a_1}[2rg(r)+(r^2-b^2)f(r)]\,\mathrm{d}r+(a_1^2-b^2)\,R_1+(b^2-a_2^2)\,R_2 \\ &\qquad +(b^2-c^2)\,(\sin\alpha_c+\psi\lambda c\cos\alpha_c)^2\,\sigma_c/(\lambda c)+2\lambda^2(\sigma_b b\cos^2\alpha_b+\sigma_c c\cos^2\alpha_c)\Big\}. \end{aligned}\right\} \quad (13{\cdot}11)$$

If the applied forces R_1, R_2, N, and the couple M are known, (13·4), (13·9), (13·10) and (13·11) furnish four relations for the determination of λ, ψ, σ_b and σ_c.

14. The symmetrical case

When the two sets of cords form a single layer and are symmetrically disposed with respect to generators of the cylindrical surface in which they lie, we may put $b=c$, $\alpha_b=-\alpha_c=\alpha$ in the results of the preceding section. Provided $\alpha\neq 0$ or $\frac{1}{2}\pi$, equations (13·2) to (13·4) then yield

$$\left.\begin{aligned} &\psi=0, \quad \mu_b^2=\mu_c^2=(1-\lambda^2\cos^2\alpha)/\sin^2\alpha, \\ &K=b^2\{\lambda(1-\lambda^2\cos^2\alpha)/\sin^2\alpha-1\}, \\ &\mu^2=\frac{b^2(1-\lambda^2\cos^2\alpha)}{r^2\sin^2\alpha}+\frac{1}{\lambda}\Big(1-\frac{b^2}{r^2}\Big), \end{aligned}\right\} \quad (14{\cdot}1)$$

and the deformation reduces to combined extension and inflation. The formulae (10·5) for $f(r)$ and $g(r)$ become

$$\left.\begin{aligned} f(r) &= 2\Big(\frac{1}{\lambda^2\mu^2}-\mu^2\Big)\Big(\frac{\partial W}{\partial I_1}+\lambda^2\frac{\partial W}{\partial I_2}\Big)\Big/(\lambda\mu^2 r), \\ g(r) &= 2\Big(\lambda^2-\frac{1}{\lambda^2\mu^2}\Big)\Big(\frac{\partial W}{\partial I_1}+\mu^2\frac{\partial W}{\partial I_2}\Big), \end{aligned}\right\} \quad (14{\cdot}2)$$

and from equations (13·9) to (13·11) we have

$$\sigma_b+\sigma_c=\lambda b\Big\{R_1-R_2-\int_{a_1}^{a_2}f(r)\,\mathrm{d}r\Big\}\Big/\sin^2\alpha, \quad (14{\cdot}3)$$

$$M=2\pi b^2\cot\alpha(1-\lambda^2\cos^2\alpha)\,(\sigma_b-\sigma_c), \quad (14{\cdot}4)$$

$$N=\frac{\pi}{\lambda}\Big\{\int_{a_2}^{a_1}[2rg(r)+(r^2+\chi_0 b^2)f(r)]\,\mathrm{d}r+(a_1^2+\chi_0 b^2)\,R_1-(a_2^2+\chi_0 b^2)\,R_2\Big\}, \quad (14{\cdot}5)$$

respectively, where $\chi_0=2\lambda^3\cot^2\alpha-1$, (14·6)

and we have used (14·3) to simplify the expression for N.

If the deformation is produced by a uniform inflating pressure P, so that $R_2=-P$ and $R_1=N=M=0$, (14·3) to (14·5) yield

$$\left.\begin{aligned} \sigma_b &= \sigma_c=\lambda b\Big\{P+\int_{a_2}^{a_1}f(r)\,\mathrm{d}r\Big\}\Big/(2\sin^2\alpha), \\ P &= -\Big\{\int_{a_2}^{a_1}[2rg(r)+(r^2+\chi_0 b^2)f(r)]\,\mathrm{d}r\Big\}\Big/(a_2^2+\chi_0 b^2). \end{aligned}\right\} \quad (14{\cdot}7)$$

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Similarly, if the deformation is produced by a tensile force N, we have $R_1 = R_2 = M = 0$, and

$$\left.\begin{aligned} \sigma_b = \sigma_c &= \lambda b \int_{a_2}^{a_1} f(r)\,\mathrm{d}r/(2\sin^2\alpha), \\ N &= \frac{\pi}{\lambda}\int_{a_2}^{a_1} \{2rg(r) + (r^2+\chi_0 b^2) f(r)\}\,\mathrm{d}r. \end{aligned}\right\} \tag{14·8}$$

The tensions in the cords are now positive if $\int_{a_2}^{a_1} f(r)\,\mathrm{d}r > 0$, and from (13·2) and (14·2) we may write

$$f(r) = 2b^2(1+\lambda\mu^2)\,(1-\lambda\mu_b^2)\left(\frac{\partial W}{\partial I_1} + \lambda^2 \frac{\partial W}{\partial I_2}\right)\bigg/(\lambda^3\mu^4 r^3).$$

If λ is positive, the sign of $f(r)$, and hence of $\int_{a_2}^{a_1} f(r)\,\mathrm{d}r$, will be that of the factor $1-\lambda\mu_b^2$. Also from (14·1)

$$1-\lambda\mu_b^2 = (1-\lambda^3)\,\{1/(\lambda^2+\lambda+1) - \cos^2\alpha\}\operatorname{cosec}^2\alpha.$$

If, therefore, α is taken to lie between 0 and $\frac{1}{2}\pi$, the tensions in the cords are positive if

$$\lambda > 1 \quad \text{and} \quad \cos^{-1}(1/\lambda) < \alpha < \cos^{-1}(\lambda^2+\lambda+1)^{-\frac{1}{2}},$$

or

$$\lambda < 1 \quad \text{and} \quad \cos^{-1}(\lambda^2+\lambda+1)^{-\frac{1}{2}} < \alpha < \tfrac{1}{2}\pi,$$

and negative if

$$\lambda > 1 \quad \text{and} \quad \cos^{-1}(\lambda^2+\lambda+1)^{-\frac{1}{2}} < \alpha < \tfrac{1}{2}\pi,$$

or

$$\lambda < 1 \quad \text{and} \quad \cos^{-1}(1/\lambda) < \alpha < \cos^{-1}(\lambda^2+\lambda+1)^{-\frac{1}{2}},$$

the lower limit for α being determined from (14·1) by the consideration that μ_b, and hence $(1-\lambda^2\cos^2\alpha)^{\frac{1}{2}}$ must be real. These conditions may be compared with the analogous results obtained in §§ 3 and 9.

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